

Dynamical Vacuum in Quantum Cosmology

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ABSTRACT

By regarding the vacuum as a perfect fluid with equation of state $p = -\rho$, de Sitter's cosmological model is quantized. Our treatment differs from previous ones in that it endows the vacuum with dynamical degrees of freedom, following modern ideas that the cosmological term is a manifestation of the vacuum energy. Instead of being postulated from the start, the cosmological constant arises from the degrees of freedom of the vacuum regarded as a dynamical entity, and a time variable can be naturally introduced. Taking the scale factor as the sole degree of freedom of the gravitational field, stationary and wave-packet solutions to the Wheeler-DeWitt equation are found, whose properties are studied. It is found that states of the Universe with a definite value of the cosmological constant do not exist. For the wave packets investigated, quantum effects are noticeable only for small values of the scale factor, a classical regime being attained at asymptotically large times.

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1. INTRODUCTION

Quantum cosmology is hopefully relevant to describe quantum gravitational effects in the very early Universe. In view of the nonexistence of a consistent quantum theory of gravity, minisuperspace quantization, which consists in “freezing out” all but a finite number of degrees of freedom of the gravitational field and its sources and quantizing the remaining ones, is expected to provide general insights on what an acceptable quantum gravity should be like. This line of attack, initiated by DeWitt [1], has been extensively pursued to quantize model universes with different symmetries and varying matter content, and allows one to conceive theories of initial conditions for the wave function of the Universe [2]. Manifold schemes have been devised to quantize gravity coupled to matter in minisuperspace, the commonest of such quantization methods being those that rely on the Wheeler-DeWitt equation, advocate the quantization of only the conformal factor of the spacetime metric, or perform canonical quantization in the reduced phase space.

In inflationary cosmology de Sitter’s model plays a fundamental role, since it describes the phase of rapid expansion during which the vacuum energy dominates the energy density of the Universe, and gives rise to a term in the energy-momentum tensor that corresponds to a cosmological constant. In modern cosmology the terms *vacuum energy* and *cosmological constant* are used almost synonymously [3]. It seems, therefore, of interest to study quantum aspects of de Sitter’s cosmological model by treating the vacuum as a dynamical entity. In such a treatment, the cosmological constant should not be postulated from the start, but should emerge from the dynamical degrees of freedom of the vacuum. A possible way to achieve this is by regarding the vacuum as a perfect fluid with equation of state $p = -\rho$. This approach appears to be fruitful, has several attractive features from the thermodynamic point of view, and leads to interesting consequences in inflationary cosmology [4]. The standard way of dealing with de Sitter’s model in quantum cosmology [5] is highly questionable because it involves a system with a single degree of freedom and one constraint, so that, strictly speaking, the system has no degrees of freedom at all and is empty of physical content. The assignation of dynamical degrees of freedom to the vacuum circumvents this difficulty and renders our method distinctive in its ability to make room for the introduction of a time variable.

Accordingly, we shall adopt Schutz’s canonical formalism [6] which describes a relativistic fluid interacting with the gravitational field. This formalism is especially adequate for our purposes, inasmuch as it has the advantage of ascribing dynamical degrees of freedom to the fluid. As it will

be seen, Schutz's action principle is successful even in the case of the vacuum in the sense that the cosmological constant appears dynamically as a manifestation of the degrees of freedom of the fluid that acts as the vacuum.

In the quantum realm the properties of de Sitter's model will be investigated on the basis of the associated Wheeler-DeWitt equation. Because the super-Hamiltonian constraint is linear in one of the momenta, the Wheeler-DeWitt equation can be reduced to a bona fide Schrödinger equation.

This paper is organized as follows. In Section 2 a Hamiltonian treatment of de Sitter's model is developed on the basis of Schutz's canonical formalism, which is proved, in the case of the vacuum, to lead to the correct classical equations of motion. In Section 3 the Wheeler-DeWitt equation is written down and is shown to take the form of a genuine Schrödinger equation for an appropriate form of the inner product. In order for the Hamiltonian operator to be self-adjoint its domain must be restricted to wave functions that obey certain boundary conditions. General sets of stationary solutions to the Wheeler-DeWitt equation obeying said boundary conditions are found. Then, in Section 4, normalized wave-packet solutions to the Wheeler-DeWitt equation are found, and their properties analyzed. Section 5 is dedicated to final comments.

2. DYNAMICAL VACUUM IN DE SITTER'S COSMOLOGICAL MODEL

The line element for a homogeneous and isotropic universe can be written in the Friedmann-Robertson-Walker form (we take $c = 1$)

$$ds^2 = g_{\nu\lambda} dx^\nu dx^\lambda = -N(t)^2 dt^2 + R(t)^2 \sigma_{ij} dx^i dx^j , \quad (2.1)$$

where σ_{ij} denotes the metric for a 3-space of constant curvature $k = +1, 0$ or -1 , corresponding to spherical, flat or hyperbolic spacelike sections, respectively.

The matter content will be taken to be a perfect fluid, and Schutz's canonical formulation of the dynamics of a relativistic fluid in interaction with the gravitational field will be employed [6]. The degrees of freedom ascribed to the fluid are five scalar potentials $\varphi, \alpha, \beta, \theta, S$ in terms of which the four-velocity of the fluid is written as

$$U_\nu = \frac{1}{\mu} (\varphi_{,\nu} + \alpha\beta_{,\nu} + \theta S_{,\nu}) , \quad (2.2)$$

where μ is the specific enthalpy. By means of the normalization condition

$$g_{\nu\lambda}U^\nu U^\lambda = -1 \quad (2.3)$$

one can express μ in terms of the velocity potentials. The action for the gravitational field plus perfect fluid is

$$S = \int_M d^4x \sqrt{-g} {}^{(4)}R + 2 \int_{\partial M} d^3x \sqrt{h} h_{ij} K^{ij} + \int_M d^4x \sqrt{-g} p \quad (2.4)$$

in units such that $c = 16\pi G = 1$. In the above equation p is the pressure of the fluid, ${}^{(4)}R$ is the scalar curvature derived from the spacetime metric $g_{\nu\lambda}$, h_{ij} is the 3-metric on the boundary ∂M of the 4-manifold M , and K^{ij} is the extrinsic curvature or second fundamental form of the boundary [7]. The surface term is necessary in the path-integral formulation of quantum gravity in order to rid the Einstein-Hilbert Lagrangian of second-order derivatives. Variations of the pressure are computed from the first law of thermodynamics.

Compatibility with the homogeneous spacetime metric is guaranteed by taking all of the velocity potentials of the fluid as functions of t only. We shall take $p = (\gamma - 1)\rho$ as equation of state for the fluid, where γ is a constant and ρ is the fluid's energy density (we shall eventually put $\gamma = 0$). In the geometry characterized by (2.1) the appropriate boundary condition for the action principle is to fix the initial and final hypersurfaces of constant time. The second fundamental form of the boundary becomes $K_{ij} = -\dot{h}_{ij}/2N$. As described in its full details in [8], after inserting the metric (2.1) into the action (2.4), using the equation of state, computing the canonical momenta and employing the constraint equations to eliminate the pair (θ, p_θ) , what remains is a reduced action in the Hamiltonian form

$$S_r = \int dt \{ \dot{R}p_R + \dot{\varphi}p_\varphi + \dot{S}p_S - N\mathcal{H} \} \quad (2.5)$$

where an overall factor of the spatial integral of $(\det \sigma)^{1/2}$ has been discarded, since it has no effect on the equations of motion. The super-Hamiltonian \mathcal{H} is given by

$$\mathcal{H} = -\frac{p_R^2}{24R} - 6kR + p_\varphi^\gamma R^{-3(\gamma-1)} e^S . \quad (2.6)$$

The lapse N plays the role of a Lagrange multiplier, and upon its variation it is found that the super-Hamiltonian \mathcal{H} vanishes. This is a constraint, revealing that the phase-space contains redundant canonical variables.

For $\gamma = 0$ the super-Hamiltonian contains neither the fluid's degree of freedom φ nor its conjugate momentum p_φ , so that these canonical variables can be simply dropped. Equivalently, the correct classical equations of motion can be obtained without taking into account the degrees of freedom described by φ, α, β and θ , that is, they could have been disregarded from the start. It is a pleasant circumstance that only the physically meaningful entropy density S is relevant for $\gamma = 0$. The action reduces to

$$S = \int dt \{ \dot{R} p_R + \dot{S} p_S - N \mathcal{H} \} \quad (2.7)$$

with

$$\mathcal{H} = -\frac{p_R^2}{24R} - 6kR + R^3 e^S . \quad (2.8)$$

This can be put in a more suggestive form by means of the canonical transformation

$$T = -e^{-S} p_S \quad , \quad p_T = e^S . \quad (2.9)$$

Then

$$S = \int dt \{ \dot{R} p_R + \dot{T} p_T - N \mathcal{H} \} \quad (2.10)$$

where

$$\mathcal{H} = -\frac{p_R^2}{24R} - 6kR + R^3 p_T . \quad (2.11)$$

The extended phase-space is generated by (R, T, p_R, p_T) . The variable T is such that the Poisson bracket

$$\{T, \mathcal{H}\}_{\mathcal{H}=0} = R^3 > 0 , \quad (2.12)$$

so that T is a “global phase time” or, more precisely, since it does not involve the canonical momenta, a “global time” in accordance with the terminology introduced by Hajicek [9]. This is reassuring because the existence of a global time appears to be a necessary condition to prevent violations of unitarity in the quantum domain.

The classical equations of motion are

$$\dot{R} = \frac{\partial(N\mathcal{H})}{\partial p_R} = -\frac{Np_R}{12R} , \quad (2.13a)$$

$$\dot{p}_R = -\frac{\partial(N\mathcal{H})}{\partial R} = N\left(-\frac{p_R^2}{24R^2} + 6k - 3R^2 p_T\right) , \quad (2.13b)$$

$$\dot{T} = \frac{\partial(N\mathcal{H})}{\partial p_T} = NR^3 , \quad (2.13c)$$

$$\dot{p}_T = -\frac{\partial(N\mathcal{H})}{\partial T} = 0 , \quad (2.13d)$$

supplemented by the super-Hamiltonian constraint

$$\mathcal{H} = -\frac{p_R^2}{24R} - 6kR + R^3 p_T = 0 \quad . \quad (2.14)$$

In order to solve these equations in the case $k = 0$ let us choose the gauge $t = T$, so that $N = R^{-3}$. Since p_T is constant, it follows from $\mathcal{H} = 0$ that p_R is proportional to R^2 . Insertion of this result in Eq.(2.13a) leads to

$$R^2 \dot{R} = \text{constant} \implies R(t) = (At)^{1/3} \quad , \quad (2.15)$$

where A is a positive constant and the origin of the time t has been conveniently chosen. The lapse function is, therefore,

$$N(t) = R^{-3} = (At)^{-1} \quad . \quad (2.16)$$

In terms of the cosmic time τ defined by

$$d\tau = N(t) dt = \frac{dt}{At} \implies \tau - \tau_0 = \frac{\ln(t)}{A} \quad (2.17)$$

one recovers the usual de Sitter solution

$$R = R_0 e^{\Lambda \tau} \quad . \quad (2.18)$$

This concludes the verification that the action (2.10) leads to de Sitter's spacetime solution to Einstein's equations. Note that if the time variable is chosen as $t = T$ the scale factor vanishes at $t = 0$, whereas in the cosmic-time gauge it vanishes only at $\tau = -\infty$. It is also worth mentioning

that Eqs.(2.9) and (2.13d) show that the entropy density S remains constant, in agreement with the behavior of inflat ionary models during the de Sitter phase [3].

The general case of arbitrary k can be easily handled in the gauge $N = 1$ and leads to the expected solutions, but we shall refrain from considering it here.

3. QUANTIZED MODEL: A WHEELER-DeWITT DESCRIPTION

It will be convenient to introduce a new parametrization of the lapse function by writing it as NR . Then the action retains the form (2.10) but the super-Hamiltonian is now

$$\mathcal{H} = -\frac{p_R^2}{24} - 6kR^2 + R^4 p_T = 0 \quad . \quad (3.1)$$

The Wheeler-DeWitt quantization scheme consists in setting

$$p_R \rightarrow -i \frac{\partial}{\partial R} \quad , \quad p_T \rightarrow -i \frac{\partial}{\partial T} \quad (3.2)$$

to form the operator $\hat{\mathcal{H}}$, and imposing the Wheeler-DeWitt equation

$$\hat{\mathcal{H}} \Psi = 0 \quad (3.3)$$

on the wave function of the universe Ψ . In our present case this equation takes the form

$$\frac{1}{24} \frac{\partial^2 \Psi}{\partial R^2} - 6kR^2 \Psi - iR^4 \frac{\partial \Psi}{\partial T} = 0 \quad . \quad (3.4)$$

Upon division by R^4 this equation takes the form of a Schrödinger equation

$$i \frac{\partial \Psi}{\partial T} = \frac{1}{24R^4} \frac{\partial^2 \Psi}{\partial R^2} - \frac{6k}{R^2} \quad (3.5)$$

with T playing the role of time. In order to be able to interpret T as a true time and (3.5) as a

genuine Schrödinger equation, the operator

$$\hat{H} = \frac{1}{24R^4} \frac{\partial^2}{\partial R^2} - \frac{6k}{R^2} \quad (3.6)$$

must be self-adjoint. The scale factor R is restricted to the domain $R > 0$, so that the minisuperspace quantization deals only with wave-functions defined on the half-line $(0, \infty)$. It is well-known that in such circumstances one has to impose boundary conditions on the allowed wave functions otherwise the relevant differential operators will not be self-adjoint. The need to impose boundary conditions to ensure self-adjointness has been long recognized by practitioners of the Arnowitt-Deser-Misner (ADM) reduced phase space formalism as applied to quantum cosmology [8,10-12], and very recently it has also been seen to have non-trivial cosmological implications in the Wheeler-DeWitt approach [13].

In the present case the operator \hat{H} given by Eq.(3.6) with $k = 0$ is self-adjoint in the inner product

$$(\psi, \phi) = \int_0^\infty R^4 \psi^*(R) \phi(R) dR \quad (3.7)$$

if its domain is suitably specified. The operator \hat{H} is symmetric if

$$(\psi, \hat{H}\phi) = \int_0^\infty \psi^*(R) \frac{d^2 \phi(R)}{dR^2} dR = \int_0^\infty \frac{d^2 \psi(R)^*}{dR^2} \phi(R) dR = (\hat{H}\psi, \phi) , \quad (3.8)$$

and, as in the case of d^2/dR^2 on $L^2(0, \infty)$, it is well known that the domain of self-adjointness of the Hamiltonian operator \hat{H} comprises only those wave functions that obey

$$\psi'(0) = \alpha \psi(0) \quad (3.9)$$

with $\alpha \in (-\infty, \infty]$. For the sake of simplicity, here we shall address ourselves in detail only to the cases $\alpha = \infty$ and $\alpha = 0$, that is, the boundary conditions we shall be mainly concerned with are

$$\Psi(0, T) = 0 \quad (3.10a)$$

or

$$\Psi'(0, T) = 0 \quad , \quad (3.10b)$$

where the prime denotes partial derivative with respect to R .

Let us look for stationary solutions to Eq.(3.4), that is, solutions of the form

$$\Psi(R, T) = e^{iET} \psi(R) \quad , \quad (3.11)$$

where E is a real parameter. Then the equation for $\psi(R)$ becomes

$$\frac{1}{24} \frac{d^2 \psi}{dR^2} + (ER^4 - 6kR^2) \psi = 0 \quad . \quad (3.12)$$

The above equation coincides with the time-independent Wheeler-DeWitt equation written by other authors, occasionally with the help of somewhat obscure methods [14], with E playing here the role of the cosmological constant Λ . It should be emphasized that here this equation has been derived from a well-defined action principle and the cosmological constant has appeared dynamically from the vacuum degrees of freedom.

In the de Sitter case ($k = 0$) it is easy to show from the above equation that the “cosmological constant” E is positive. Indeed, multiplying Eq.(3.12) by ψ^* and integrating over the half-line one finds

$$- \int_0^\infty \psi^*(R) \frac{d^2 \psi(R)}{dR^2} dR = 24E \int_0^\infty R^4 |\psi(R)|^2 dR \quad (3.13)$$

which, after an integration by parts followed by the use of (3.9), yields, for $\alpha \geq 0$,

$$E = \frac{1}{24} \frac{\alpha |\psi(0)|^2 + \int_0^\infty |d\psi/dR|^2 dR}{\int_0^\infty R^4 |\psi(R)|^2 dR} > 0 \quad (3.14)$$

as we wished to prove. It should be clear from the above derivation that the general boundary condition (3.9) is not sufficient to allow us to reach the same conclusion. This special property of

conditions (3.9) with $\alpha \geq 0$ is not present in other minisuperspace models, and seems to confer this restricted set of boundary conditions a physically privileged status as compared to the general one with arbitrary α .

The general solution to Eq.(3.12) with $k = 0$ is [15]

$$\psi_E(R) = \sqrt{R} \left[A J_{1/6}(\beta R^3/3) + B J_{-1/6}(\beta R^3/3) \right] , \quad (3.15)$$

where J_ν denotes a Bessel function of the first kind and order ν , A and B are arbitrary constants, and

$$\beta = \sqrt{24E} . \quad (3.16)$$

The usual interpretation of $R^4|\Psi|^2$ as a probability density implies no correlation between R and T . The existence of such solutions to the Wheeler-DeWitt equation is perhaps not surprising since de Sitter's spacetime may be regarded as static [16] or self-similar [17].

It follows from the behavior of Bessel functions for small argument that in the case of boundary condition (3.10a) the solution is

$$\psi_E^{(a)}(R) = \sqrt{R} J_{1/6}(\beta R^3/3) , \quad (3.17a)$$

whereas in the case of boundary condition (3.10b) the solution is

$$\psi_E^{(b)}(R) = \sqrt{R} J_{-1/6}(\beta R^3/3) . \quad (3.17b)$$

From the asymptotic behavior of Bessel functions for small and large argument one easily checks that both solutions are square integrable, but their norm induced by the inner product (3.7) is infinite. Thus, states of the Universe with a definite value of the cosmological constant do not exist. Realizable states can only be constructed by superposition of solutions to the Wheeler-DeWitt equation with different values of the cosmological constant.

For any two states ψ_1 and ψ_2 belonging to the domain of the Hamiltonian operator, that is, obeying condition (3.9), one has

$$J_{12}(0) = \frac{i}{2} \left(\psi_1^* \frac{\partial \psi_2}{\partial R} - \psi_2 \frac{\partial \psi_1^*}{\partial R} \right)_{R=0} = 0 . \quad (3.18)$$

Therefore, as in other minisuperspace models [12,18], here Vilenkin's wave function of the universe Ψ is ruled out because it is in conflict with the self-adjointness of the Hamiltonian operator. Indeed, Vilenkin's tunneling boundary condition [2,5] requires the wave function of the universe Ψ to consist only of outgoing modes at singular boundaries of superspace. In the present context this would amount mathematically to $J_{12}(0) > 0$ whenever $\psi_1 = \psi_2 = \Psi$, which is impossible.

4. EVOLUTION OF WAVE PACKETS

The stationary solutions (3.17) have infinite norm and play here a role analogous to that of plane waves in the quantum mechanics of the free particle, that is, finite-norm solutions can be constructed by superposing them. The general solutions to the Wheeler-DeWitt equation (3.5) with $k = 0$ are given by the continuous linear combinations

$$\Psi^{(\sigma)}(R, T) = \int_0^\infty c^{(\sigma)}(E) e^{iET} \psi_E^{(\sigma)}(R) \ , \quad \sigma = a, b \ , \quad (4.1)$$

where the superscript is used to distinguish the wave functions that obey the boundary condition (3.10a) from those that obey (3.10b). According to the Appendix, the probability distribution of values of the cosmological constant is given by

$$\rho^{(\sigma)}(E) = \frac{1}{4} |c^{(\sigma)}(E)|^2 \ , \quad (4.2)$$

assuming, of course, that $\Psi^{(\sigma)}(R, T)$ is normalized in the inner product (3.7).

We shall consider simple but illustrative examples of wave-packet solutions to the Wheeler-DeWitt equation obeying each of the boundary conditions (3.10). Introducing the parameter

$$\lambda = \frac{\sqrt{24E}}{3} \quad (4.3)$$

we can write (4.1) as

$$\Psi^{(\sigma)}(R, T) = \sqrt{R} \int_0^\infty a^{(\sigma)}(\lambda) e^{i3\lambda^2 T/8} J_\nu(\lambda R^3) d\lambda \quad (4.4)$$

where $\nu = +1/6$ or $\nu = -1/6$ according to whether $\sigma = a$ or b , and

$$a^{(\sigma)}(\lambda) = \frac{3\lambda}{4} c^{(\sigma)}\left(\frac{3\lambda^2}{8}\right). \quad (4.5)$$

.

The choice

$$a^{(\sigma)}(\lambda) = \lambda^{\nu+1} e^{-\alpha\lambda^2} \quad , \quad \alpha > 0 \quad , \quad (4.6)$$

with $\nu = +1/6$ for $\sigma = a$ and $\nu = -1/6$ for $\sigma = b$, is particularly simple because it enables us to perform the integration in (4.4) and express the wave function of the Universe in terms of elementary functions [19]. In the case $\nu = 1/6$ we find

$$\Psi^{(a)}(R, T) = \left[2\left(\alpha - \frac{3iT}{8}\right)\right]^{-7/6} R \exp\left(-\frac{R^6}{4\left(\alpha - \frac{3iT}{8}\right)}\right) \quad , \quad (4.7)$$

whereas for $\nu = -1/6$ the result is

$$\Psi^{(b)}(R, T) = \left[2\left(\alpha - \frac{3iT}{8}\right)\right]^{-5/6} \exp\left(-\frac{R^6}{4\left(\alpha - \frac{3iT}{8}\right)}\right) \quad . \quad (4.8)$$

The expectation value of the scale factor is given by

$$\langle R \rangle_T = \frac{\int_0^\infty R^4 \Psi^*(R, T) R \Psi(R, T) dR}{\int_0^\infty R^4 \Psi^*(R, T) \Psi(R, T) dR} \quad , \quad (4.9)$$

and its time dependence reflects the dynamical evolution of the quantized version of de Sitter's cosmological model. For the two types of boundary conditions of interest we find, respectively,

$$\langle R \rangle_T^{(a)} = \frac{\Gamma(\frac{4}{3})}{\Gamma(\frac{7}{6})} \left[\frac{64\alpha^2 + 9T^2}{32\alpha} \right]^{1/6} \quad , \quad (4.10)$$

and

$$\langle R \rangle_T^{(b)} = \frac{1}{\Gamma(\frac{5}{6})} \left[\frac{64\alpha^2 + 9T^2}{32\alpha} \right]^{1/6} \quad . \quad (4.11)$$

For sufficiently large values of the time T the expectation value $\langle R \rangle_T$ grows at the same rate predicted by the classical solution (2.15), that is, the classical regime is attained for asymptotically large times. Quantum effects make themselves felt only for small enough T corresponding to small R , as expected.

The dispersion of the wave packets defined by

$$(\Delta R)_T^2 = \langle R^2 \rangle_T - \langle R \rangle_T^2 \quad (4.12)$$

is readily computed, with the results

$$(\Delta R)_T^{(a)} = \left(\frac{\Gamma(3/2)}{\Gamma(7/6)} - \frac{\Gamma(4/3)^2}{\Gamma(7/6)^2} \right)^{1/2} \left[\frac{64\alpha^2 + 9T^2}{32\alpha} \right]^{1/6} \quad , \quad (4.13a)$$

$$(\Delta R)_T^{(b)} = \left(\frac{\Gamma(5/6)\Gamma(7/6) - 1}{\Gamma(5/6)^2} \right)^{1/2} \left[\frac{64\alpha^2 + 9T^2}{32\alpha} \right]^{1/6} \quad . \quad (4.13b)$$

The wave packets inevitably disperse as time passes, the minimum width being attained at $T = 0$. As in the case of the free particle, the more localized the initial state at $T = 0$ the more rapidly the wave packet disperses.

It is important to classify the nature of this model as concerns the presence or absence of singularities. For the states (4.7) and (4.8) the expectation value of R never vanishes, showing that these states are nonsingular. The issue of existence or nonexistence of singularities may be addressed from another point of view [20]. We can define the probability density

$$P^{(\sigma)}(R) = R^4 |\Psi_E^{(\sigma)}(R)|^2 \quad , \quad \sigma = a, b \quad , \quad (4.14)$$

for the stationary solutions (3.17a) and (3.17b). The behavior of the Bessel functions for small values of the argument makes it clear that $P^{(\sigma)}(R) \rightarrow 0$ as $R \rightarrow 0$, and thus the singularity is avoided within this mode 1 according to this criterion. Whatever the singularity criterion, de Sitter's quantum cosmological model is nonsingular just as its classical counterpart.

5. CONCLUSION

We have shown that taking the vacuum as a perfect fluid with equation of state $p = -\rho$ a Hamiltonian description of de Sitter's cosmological model is possible, which makes subsequent quantization a straightforward process. This circumvents the problem of insufficient number of degrees of freedom that besets the usual Wheeler-DeWitt quantization of de Sitter's model. The endowment of the vacuum with dynamical degrees of freedom makes it possible the introduction of a time variable which, in turn, gives meaning to the dynamical evolution at the quantum level. The cosmological term is not postulated from the beginning, but arises as a manifestation of the vacuum degrees of freedom. In our approach states with a definite value of the cosmological constant are ruled out, and only those states are realizable that are finite-norm superpositions of solutions to the Wheeler-DeWitt equation with different values of the cosmological constant.

With the scale factor as the sole degree of freedom of the gravitational field, stationary and simple wave-packet solutions to the Wheeler-DeWitt equation have been found. It turns out that, for the wave packets investigated, quantum effects are significant only for small values of the scale

factor, and the classical regime sets in at asymptotically large times. Just like the classical de Sitter model, its quantum counterpart is nonsingular.

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APPENDIX: COSMOLOGICAL CONSTANT PROBABILITY DENSITY

Let us start with a normalized state vector $\Psi(R, t)$ for which

$$\|\Psi\|^2 = \int_0^\infty R^4 |\Psi(R, t)|^2 dR = \int_0^\infty dR R^4 \int_0^\infty c(E)^* e^{-iEt} \sqrt{R} J_\nu(\sqrt{24E} R^3/3) dE \int_0^\infty c(E') e^{iE't} \sqrt{R} J_\nu(\sqrt{24E'} R^3/3) dE' . \quad (A.1)$$

The change of variables

$$E = \frac{9}{24} \lambda^2 \quad , \quad E' = \frac{9}{24} \lambda'^2 \quad , \quad x = R^3 \quad , \quad g(\lambda) = c(9\lambda^2/24) \exp(-i9\lambda^2 t/24) \quad , \quad (A.2)$$

leads to

$$\|\Psi\|^2 = \frac{3}{16} \int_0^\infty d\lambda \lambda g(\lambda)^* \int_0^\infty d\lambda' \lambda' g(\lambda') \int_0^\infty x J_\nu(\lambda x) J_\nu(\lambda' x) dx . \quad (A.3)$$

With the help of Hankel's integral formula [21]

$$f(x) = \int_0^\infty J_\nu(tx) t dt \int_0^\infty f(\lambda) J_\nu(\lambda t) \lambda d\lambda \quad , \quad (A.4)$$

which is equivalent to the formal equation

$$\int_0^\infty x J_\nu(\lambda x) J_\nu(\lambda' x) dx = \frac{1}{\lambda} \delta(\lambda - \lambda') \quad , \quad (A.5)$$

one finds

$$\|\Psi\|^2 = \frac{3}{16} \int_0^\infty d\lambda \lambda |g(\lambda)|^2 = \frac{3}{16} \int_0^\infty d\lambda \lambda |c(\frac{9\lambda^2}{24})|^2 = \frac{1}{4} \int_0^\infty |c(E)|^2 dE \quad , \quad (A.6)$$

from which Eq.(4.2) follows.

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